

MST204

Assignment Booklet I

| Contents | Cut-off date |
|---|------------------|
| 2 TMA MST204 01 Part 1 (covering the <i>Preparatory Unit</i>) | 5 February 1998 |
| 3 TMA MST204 01 Part 2 (covering <i>Units 1, 2 and 3</i>) | 26 February 1998 |
| 6 CMA MST204 41 (covering <i>Units 1, 2, 3 and 4</i>) | 5 March 1998 |
| 12 CMA MST204 42 (covering <i>Units 5, 6 and 7</i>) | 26 March 1998 |
| 16 TMA MST204 02 (covering <i>Units 4, 6, 7 and 8</i>) | 9 April 1998 |
| 19 CMA MST204 43 (covering <i>Units 8, 9, 12 and 14</i>) | 30 April 1998 |

The marks allocated to each part of each TMA question are indicated in brackets in the margin.

Please send all your answers to each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above. Remember to fill in the correct *Assignment Number* as listed above and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

Similarly, each completed computer-marked assignment (CMA) form should be posted in an envelope marked *Computer-marked Assignment* so as to arrive at the Open University by the appropriate cut-off date.

Question 1 below, on the *Preparatory Unit*, forms the first part of Tutor-marked Assignment MST204 01. The remainder of the TMA (Part 2, on *Units 1, 2 and 3*) can be found immediately following Question 1 in this booklet. Question 1 is marked out of 25. (The whole TMA is marked out of 100.)

Please send your answers to Question 1 to your tutor, along with an assignment form (PT3). Be sure to fill in the Assignment Number on this form as

MST204 01

Your tutor will mark and comment on your solution to Question 1, and will send your script back to you directly to give you some early feedback on the course. He or she will retain your PT3 to enter your marks for the rest of this assignment on it. The form will then be sent to you via Walton Hall, so that your mark can be recorded.

Question 1 (*Unit P*)

- (a) Make the variable y the subject of the equation

$$\log_e y - \log_e(y+3) + \log_e 4 = 2 \log_e x. \quad [6]$$

- (b) By using *implicit differentiation*, find the slope of the tangent to the curve

$$x^3 - 2xy + y^2 = 1$$

at the point $(1, 2)$. Hence find the equation of the tangent to the curve at this point. [6]

- (c) (i) Write

$$\frac{A}{x} + \frac{B}{x+2},$$

where A and B are constants, in the form

$$\frac{\text{expression}}{x(x+2)}. \quad [2]$$

- (ii) Find values of A and B such that

$$\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}. \quad [2]$$

- (iii) Hence evaluate

$$\int \frac{1}{x(x+2)} dx \quad (x > 0). \quad [2]$$

- (d) Evaluate the following integrals.

(i) $\int \frac{v^3}{v^4 + 5} dv \quad [3]$

(ii) $\int_2^5 \frac{t}{\sqrt{t-1}} dt \quad [4]$

Questions 2, 3 and 4 below, on *Units 1, 2 and 3*, form the second part of Tutor-marked Assignment MST204 01.

Each question is marked out of 25. Your overall grade will be based on the sum of your marks on these three questions and the question in Part 1.

Please send your answers to Questions 2 to 4 to your tutor. Your tutor should have kept the PT3 for this assignment so that there is no need to send another. (If your tutor has returned your original PT3 by mistake with your answer to Question 1, send it back with your answers to Questions 2 to 4.) You will eventually receive your copy of this PT3, completed by your tutor, along with your answers to these questions.

Question 2 (*Unit 1*)

- (i) Find the general solution of the recurrence relation

$$u_{r+1} = 7u_r + 30u_{r-1}. \quad [5]$$

Hence determine the particular solution of the recurrence relation which satisfies the initial conditions

$$u_0 = \frac{1}{3} \quad \text{and} \quad u_1 = -1.$$

Calculate the exact value of u_{10} using your solution. [5]

- (ii) On your calculator, compute the values of u_2, u_3, \dots, u_{10} using the recurrence relation in part (i) with 0.3333 as an approximation to the value of u_0 . [7]

- (iii) By examining the absolute and relative errors in the values of u_{10} resulting from the change from the exact value of u_0 given in part (i) to the approximation for u_0 used in part (ii), decide whether or not the problem of calculating u_{10} using the recurrence relation above with the initial conditions $u_0 = \frac{1}{3}$ and $u_1 = -1$ is

- absolutely ill-conditioned,
- relatively ill-conditioned,

with respect to small changes in the value of u_0 , giving the reasons for your conclusions. [8]

Question 3 (Unit 2)

- (a) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x(x+3)} \quad (x > 0)$$

which satisfies the condition $y = 2$ when $x = 1$, expressing your answer in the form $y = f(x)$.

[9]

[You may assume that the required solution satisfies $y > 0$.]

- (b) Find the particular solution of the differential equation

$$x \frac{dy}{dx} = 4y + 3x^7 \quad (x > 0)$$

which satisfies the condition $y = 3$ when $x = 1$.

[8]

- (c) This part of the question is concerned with the use of Euler's method to find an approximate solution of the differential equation

$$\frac{dy}{dx} = x^2 - y^2 \quad (x \geq 0)$$

with initial condition $y(1) = 0$.

- (i) Write down the recurrence relation for Euler's numerical method applied to the above problem. Use this recurrence relation with a step length $h = 0.1$ to calculate approximations to $y(1.1)$ and $y(1.2)$.

[4]

- (ii) The values shown in the table below were obtained for the approximation to $y(1.2)$ given by Euler's method with various step lengths h , using a computer. Plot a graph of these approximations to $y(1.2)$ against step length h .

| Step length h | Approximation to $y(1.2)$ |
|-----------------|---------------------------|
| 0.001 | 0.238 964 |
| 0.002 | 0.238 778 |
| 0.004 | 0.238 406 |
| 0.005 | 0.238 220 |
| 0.008 | 0.237 661 |
| 0.01 | 0.237 287 |

What does your graph indicate to be the relationship between the Euler approximation to $y(1.2)$ and the step length h for small h ? Use your graph to estimate the true value of $y(1.2)$ correct to five decimal places.

[4]

Question 4 (Unit 3)

The laboratory culture of bacteria is carried out in shallow dishes containing a suitable growing medium; the area occupied by the bacteria at any time is a convenient measure of the size of the population of bacteria. This question is concerned with setting up a model of the growth of a bacterial colony using the logistic equation.

The following table gives data from an experiment to investigate the growth of a bacterial colony in the laboratory.

| Age of colony (days) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------------------|------|------|------|-------|-------|-------|-------|
| Area (cm ²) | 1.00 | 3.15 | 8.90 | 19.81 | 31.40 | 38.13 | 40.75 |

The model will describe the development of any culture of bacteria of the type used in the experiment: the experimental data serve to fix the parameters in the model.

- (i) Let A_n denote the area of the experimental colony (in cm²) on the n th day. Make a plot of this quantity against n . Using your graph, explain why the logistic equation is likely to provide a suitable model for the growth of the bacteria.

[3]

Assume that the growth in area of a bacterial colony obeys the logistic equation

$$\frac{dA}{dt} = a \left(1 - \frac{A}{M} \right) A,$$

where A is the area in cm² and t the time in days.

- (ii) By plotting $(A_{n+1} - A_n)/A_n$ against A_{n+1} , estimate the parameters a and M , using the method outlined in Exercise 4 on page 24 of *Unit 3*.

[6]

- (iii) What will be the limiting value of the area of a culture of this type of bacteria if the colony is left to grow indefinitely? How long will it take for the culture to reach 99% of this limiting area?

[4]

A simple system for the daily production of bacteria for biological experiments works as follows. A certain quantity of the bacteria is placed in a growing dish, and exactly 1 day later the amount generated in that time is 'harvested', so that the next day begins with the quantity of bacteria in the dish the same as on the first day. This process can then be repeated day by day.

- (iv) Show that the increase I in the amount of bacteria in a day depends on the initial amount A_{in} (i.e. the amount present at the beginning of the day) according to the formula

$$I = \frac{Me^a A_{in}}{M + (e^a - 1)A_{in}} - A_{in}.$$

[2]

- (v) Using the formula from part (iv), show that the initial amount of bacteria which will give the maximum daily yield is

$$A_{in} = \frac{M}{e^{a/2} + 1}.$$

[8]

- (vi) For the values of the parameters a and M you found in part (ii), find the maximum daily yield and the initial amount of bacteria which will give this yield.

[2]

This assignment covers *Units 1, 2, 3 and 4*.

Unit 1

Question 1

Select the TWO options which correctly describe the recurrence relation

$$u_{r+1} = 3ru_r - (r-1)u_{r-1}.$$

Options

- A First-order and linear
- B First-order and non-linear
- C Second-order and linear
- D Second-order and non-linear
- E Constant coefficients and homogeneous
- F Constant coefficients and non-homogeneous
- G Non-constant coefficients and homogeneous
- H Non-constant coefficients and non-homogeneous

[There are TWO correct options.]

Question 2

Select the option which is the general solution of the recurrence relation

$$u_{r+1} = 3 - 2u_r.$$

Options

- | | | |
|-----------------------|-----------------------|-----------------------|
| A $u_n = -A(2)^n + 1$ | B $u_n = -A(2)^n - 1$ | C $u_n = -A(2)^n + 3$ |
| D $u_n = -A(2)^n - 3$ | E $u_n = A(-2)^n + 1$ | F $u_n = A(-2)^n - 1$ |
| G $u_n = A(-2)^n + 3$ | H $u_n = A(-2)^n - 3$ | |

Question 3

Select the option which is the general solution of the recurrence relation

$$u_{r+1} = u_r - 4.$$

Options

- | | | |
|---------------------------------|---------------------------------|-----------------------|
| A $u_n = A + 4n$ | B $u_n = A - 4n$ | C $u_n = An + 4$ |
| D $u_n = An - 4$ | E $u_n = B(-4)^n + 2$ | F $u_n = B(-4)^n - 2$ |
| G $u_n = B(-4)^n + \frac{1}{5}$ | H $u_n = B(-4)^n - \frac{1}{5}$ | |

Question 4

Select the option which is the general solution of the recurrence relation

$$u_{r+1} = 8u_r - 16u_{r-1}.$$

Options

- | | |
|-----------------------------|------------------------------|
| A $u_n = A(4)^n + B(4)^n$ | B $u_n = A(4)^n + Bn(4)^n$ |
| C $u_n = A(-4)^n + B(-4)^n$ | D $u_n = A(-4)^n + Bn(-4)^n$ |
| E $u_n = A(4)^n + B(-4)^n$ | F $u_n = A(4)^n + Bn(-4)^n$ |
| G $u_n = An(4)^n + B(-4)^n$ | H $u_n = An(4)^n + Bn(-4)^n$ |
-

Question 5

Select the option which is TRUE for the problem of calculating u_n , where n is large, by using the recurrence relation

$$u_{r+1} = \frac{3}{2} - \frac{1}{2}u_r$$

with the initial condition $u_0 = 1$.

Options

- A The problem is absolutely and relatively well-conditioned with respect to small changes in the value of u_0 .
 - B The problem is absolutely well-conditioned and relatively ill-conditioned with respect to small changes in the value of u_0 .
 - C The problem is absolutely ill-conditioned and relatively well-conditioned with respect to small changes in the value of u_0 .
 - D The problem is absolutely and relatively ill-conditioned with respect to small changes in the value of u_0 .
-

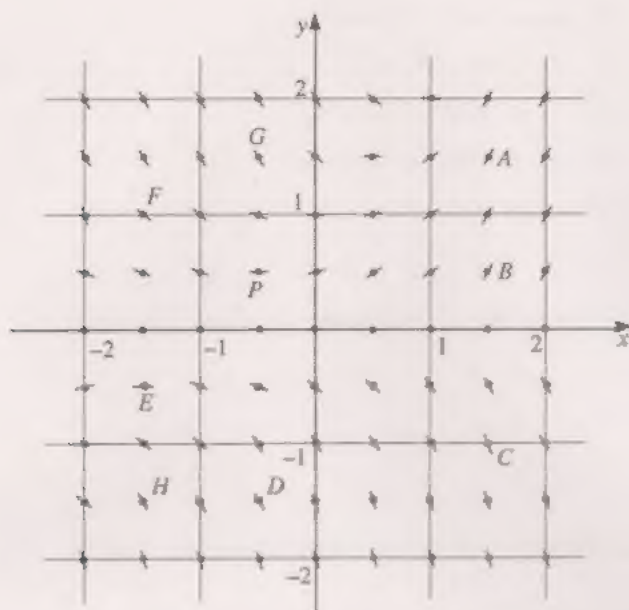
Question 6

Using the notation and model introduced in the television programme associated with *Unit 1*, select the option which models the mortgage problem in which £30 000 is borrowed, the gross repayments (before any income tax relief) are £300 per month, and the interest rate is 10% per annum.

Options

- A $u_0 = 30\,000$ and $u_{r+1} = 0.1u_r - 300$
 - B $u_0 = 30\,000$ and $u_{r+1} = 0.9u_r - 300$
 - C $u_0 = 30\,000$ and $u_{r+1} = 1.1u_r - 300$
 - D $u_0 = 30\,000$ and $u_{r+1} = 0.1u_r - 3600$
 - E $u_0 = 30\,000$ and $u_{r+1} = 0.9u_r - 3600$
 - F $u_0 = 30\,000$ and $u_{r+1} = 1.1u_r - 3600$
-

Questions 7 and 8



- 7 For the direction field shown above, sketch the trajectory which passes through the point P . Select which TWO of the points marked A, B, C, \dots, H , are closest to this trajectory.

Options

- A Point A B Point B C Point C D Point D
E Point E F Point F G Point G H Point H

[There are TWO correct options.]

- 8 Select the differential equation which is the one most likely to be satisfied by the trajectories of the direction field shown above.

Options

- A $\frac{dy}{dx} = x(x - y + 1)$ B $\frac{dy}{dx} = x(y - x - 1)$ C $\frac{dy}{dx} = y(x - y + 1)$
D $\frac{dy}{dx} = y(y - x - 1)$ E $\frac{dy}{dx} = \frac{x}{x - y + 1}$ F $\frac{dy}{dx} = \frac{x}{y - x - 1}$
G $\frac{dy}{dx} = \frac{y}{x - y + 1}$ H $\frac{dy}{dx} = \frac{y}{y - x - 1}$

Questions 9 to 11

For each of the following differential equations, select the option which describes the possible method or methods of solution, assuming that the necessary integrations can be performed.

9 $\frac{dy}{dx} = x(2 + y)$

10 $\frac{dy}{dx} = x(x^2 + y)$

11 $\frac{dy}{dx} = y(x^2 + y)$

Options for Questions 9 to 11

- A Neither the separation of variables method nor the integrating factor method
 - B The separation of variables method but not the integrating factor method
 - C The integrating factor method but not the separation of variables method
 - D Either the separation of variables method or the integrating factor method
-

Question 12

Select the option which gives an integrating factor for the differential equation

$$\frac{dy}{dx} = \cos x + y \sin x.$$

Options

- | | | | |
|-----------------|----------------|-----------------|----------------|
| A $-\cos x$ | B $\cos x$ | C $-\sin x$ | D $\sin x$ |
| E $e^{-\cos x}$ | F $e^{\cos x}$ | G $e^{-\sin x}$ | H $e^{\sin x}$ |
-

Unit 3

Questions 13 to 16

A continuous (i.e. differential equation) model of the way a population changes with time is constructed using the following assumptions.

- The proportionate birth rate is a decreasing linear function of population size P . When the population was 400, the proportionate birth rate was 35% per year; when the population was 600, the proportionate birth rate was 30% per year.
- The proportionate death rate is constant at 25% per year.
- There is no migration or exploitation of the population.

- 13 Select the option which gives the correct expression for the annual proportionate birth rate.

Options

- | | |
|------------------------|------------------------|
| A $0.45 + 0.00025P$ | B $0.45 - 0.00025P$ |
| C $45 + 0.025P$ | D $45 - 0.025P$ |
| E $(0.45 + 0.00025P)P$ | F $(0.45 - 0.00025P)P$ |
| G $(45 + 0.025P)P$ | H $(45 - 0.025P)P$ |

- 14 Select the option which gives the differential equation which models the development of this population, where t is time measured in years.

Options

- | | |
|---------------------------------------|---------------------------------------|
| A $\frac{dP}{dt} = 0.2 + 0.00025P$ | B $\frac{dP}{dt} = 0.2 - 0.00025P$ |
| C $\frac{dP}{dt} = 20 + 0.025P$ | D $\frac{dP}{dt} = 20 - 0.025P$ |
| E $\frac{dP}{dt} = (0.2 + 0.00025P)P$ | F $\frac{dP}{dt} = (0.2 - 0.00025P)P$ |
| G $\frac{dP}{dt} = (20 + 0.025P)P$ | H $\frac{dP}{dt} = (20 - 0.025P)P$ |

- 15 Select the option which is closest to the value of the parameter M (in the notation of *Unit 3*) of the logistic model for this population.

Options

- | | | | |
|--------|--------|--------|--------|
| A 200 | B 400 | C 600 | D 800 |
| E 1000 | F 1200 | G 1400 | H 1600 |

- 16 Select the option which is closest to the value of the parameter a (in the notation of *Unit 3*), measured in units of year^{-1} , of the logistic model for this population.

Options

- | | | | |
|-------|--------|---------|--------|
| A -45 | B -20 | C -0.45 | D -0.2 |
| E 0.2 | F 0.45 | G 20 | H 45 |

Questions 17 and 18

The growth of a population P with time t (measured in years), when not exploited, may be described by the differential equation

$$\frac{dP}{dt} = 0.4P \left(1 - \frac{P}{1000} \right).$$

It is now proposed to exploit the population by taking a fixed proportion of the population each year.

- 17 Select the option which is the proportion of the population which should be taken per year in order to give the maximum sustainable yield.

Options

- | | | | |
|-------|-------|-------|-------|
| A 10% | B 20% | C 30% | D 40% |
| E 50% | F 60% | G 70% | H 80% |

- 18 Select the option which is the annual maximum sustainable yield from the population.

Options

- | | | | |
|-------|-------|-------|--------|
| A 100 | B 160 | C 200 | D 320 |
| E 400 | F 640 | G 800 | H 1600 |

Unit 4

Questions 19 to 24

A particle of mass $m = \frac{1}{2}$ moves along a chosen x -axis so that its position at time t is given by

$$x = 8 - 4t + 2t^2$$

All variables are measured in the appropriate SI units.

- 19 Select the option which is the velocity of the particle at the instant $t = 1$.
- 20 Select the option which is the speed of the particle at the instant $t = 1$.
- 21 Select the option which is the acceleration of the particle at the instant $t = 1$.
- 22 Select the option which is the x -component of the total force acting on the particle at the instant $t = 1$.
- 23 Select the option which is the magnitude of the total force acting on the particle at the instant $t = 1$.

Options for Questions 19 to 23

- | | | | | | | | |
|---|----|---|----|---|----|---|----|
| A | -8 | B | -6 | C | -4 | D | -2 |
| E | 2 | F | 4 | G | 6 | H | 8 |

- 24 Either select the option which describes the direction of the total force acting on the particle at the instant $t = 1$, or select option C if you think that there is no total force acting on the particle at this instant.

Options

- A The total force is in the direction of increasing x .
 - B The total force is in the direction of decreasing x .
 - C There is no total force acting on the particle at this instant.
-

This assignment covers *Units 5, 6 and 7*.

Unit 5

Questions 1 to 3

Consider the complex numbers

$$z_1 = -1 + 2i \quad \text{and} \quad z_2 = 4 - 3i.$$

- 1 Select the option which is z_1^2 .
- 2 Select the option which is z_2/z_1 .

Options for Questions 1 and 2

- A $-2 - i$ B $-2 + i$ C $-1 - 2i$ D $-1 + 2i$
 E $1 - 2i$ F $1 + 2i$ G $2 - i$ H $2 + i$

- 3 Select the option which is $|z_2|$.

Options

- A 1 B 3 C 4 D 5
 E 7 F 16 G 25 H 49

Question 4

Select the option which is the complex number with modulus 2 and argument $-\frac{1}{3}\pi$.

Options

- A $1 + \sqrt{3}i$ B $1 - \sqrt{3}i$ C $-1 + \sqrt{3}i$ D $-1 - \sqrt{3}i$
 E $\sqrt{3} + i$ F $\sqrt{3} - i$ G $-\sqrt{3} + i$ H $-\sqrt{3} - i$

Questions 5 and 6

Consider the complex number $z = -2\sqrt{3} + 2i$.

- 5 Select the option which is the modulus of z .

Options

- A 2 B 4 C 8 D 12
 E 16 F 32 G 48 H $2\sqrt{3}$

- 6 Select the option which is the argument of z .

Options

- A $-\frac{5}{6}\pi$ B $-\frac{2}{3}\pi$ C $-\frac{1}{3}\pi$ D $-\frac{1}{6}\pi$
 E $\frac{1}{6}\pi$ F $\frac{1}{3}\pi$ G $\frac{2}{3}\pi$ H $\frac{5}{6}\pi$

Question 7

Select the option which is the exponential form of the complex number $-2 + 2i$.

Options

- A $2e^{-i3\pi/4}$ B $2e^{-i\pi/4}$ C $2e^{i\pi/4}$ D $2e^{i3\pi/4}$
 E $2\sqrt{2}e^{-i3\pi/4}$ F $2\sqrt{2}e^{-i\pi/4}$ G $2\sqrt{2}e^{i\pi/4}$ H $2\sqrt{2}e^{i3\pi/4}$
-

Questions 8 and 9

For any θ , $\sin^5 \theta$ may be expressed in the form $a \sin 5\theta + b \sin 3\theta + c \sin \theta$, where a , b and c are real numbers.

- 8 What is the value of a ?
 9 What is the value of c ?

Options for Questions 8 and 9

- A $\frac{1}{16}$ B $\frac{1}{8}$ C $\frac{5}{16}$ D $\frac{1}{4}$
 E $-\frac{1}{16}$ F $-\frac{1}{8}$ G $-\frac{5}{16}$ H $-\frac{1}{4}$
-

Question 10

Select the option which is the phasor of the sinusoidal function

$$3 \cos \omega t - 5 \sin \omega t,$$

where ω is a positive constant.

Options

- A $-5 - 3i$ B $-5 + 3i$ C $-3 - 5i$ D $-3 + 5i$
 E $3 - 5i$ F $3 + 5i$ G $5 - 3i$ H $5 + 3i$
-

Question 11

Select the option which is the sinusoidal function of period $2\pi/\omega$ whose phasor is $1 - i$.

Options

- A $2 \cos(\omega t + \frac{3}{4}\pi)$ B $2 \cos(\omega t + \frac{1}{4}\pi)$ C $2 \cos(\omega t - \frac{1}{4}\pi)$
 D $2 \cos(\omega t - \frac{3}{4}\pi)$ E $\sqrt{2} \cos(\omega t + \frac{3}{4}\pi)$ F $\sqrt{2} \cos(\omega t + \frac{1}{4}\pi)$
 G $\sqrt{2} \cos(\omega t - \frac{1}{4}\pi)$ H $\sqrt{2} \cos(\omega t - \frac{3}{4}\pi)$
-

Question 12

Select the option which is an expression for the general solution of the recurrence relation

$$u_{r+1} = -2\sqrt{2}u_r - 4u_{r-1}.$$

Options

- A $u_n = 2^{n/2} (A \cos \frac{1}{4}n\pi + B \sin \frac{1}{4}n\pi)$ B $u_n = 2^{n/2} (A \cos \frac{3}{4}n\pi + B \sin \frac{3}{4}n\pi)$
 C $u_n = 2^n (A \cos \frac{1}{4}n\pi + B \sin \frac{1}{4}n\pi)$ D $u_n = 2^n (A \cos \frac{3}{4}n\pi + B \sin \frac{3}{4}n\pi)$
 E $u_n = 8^{n/2} (A \cos \frac{1}{4}n\pi + B \sin \frac{1}{4}n\pi)$ F $u_n = 8^{n/2} (A \cos \frac{3}{4}n\pi + B \sin \frac{3}{4}n\pi)$
 G $u_n = 4^n (A \cos \frac{1}{4}n\pi + B \sin \frac{1}{4}n\pi)$ H $u_n = 4^n (A \cos \frac{3}{4}n\pi + B \sin \frac{3}{4}n\pi)$
-

Questions 13 to 16

For each of the following differential equations, select the option which gives an expression for its general solution.

13 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

14 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

15 $\frac{d^2y}{dx^2} + 4y = 0$

16 $\frac{d^2y}{dx^2} - 4y = 0$

Options for Questions 13 to 16

A $y = Ae^{-2x}$

B $y = Ae^{2x}$

C $y = A + Be^{-2x}$

D $y = A + Be^{2x}$

E $y = Ae^{-2x} + Be^{2x}$

F $y = Ae^{-2x} + Bxe^{-2x}$

G $y = Ae^{2x} + Bxe^{2x}$

H $y = A \cos 2x + B \sin 2x$

Questions 17 and 18

The complementary function of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = f(x)$$

is

$$y = Ae^x + Be^{3x}.$$

In each of the following cases, select the option which is the BEST function to use as a trial solution in the method described in *Unit 6* for finding a particular solution of the differential equation.

17 $f(x) = 2e^{3x}$

Options

A $2e^{3x}$

B ae^{3x}

C axe^{4x}

D $(ax + b)e^{3x}$

E ax^2e^{3x}

F $(ax^2 + b)e^{3x}$

G $(ax^2 + bx)e^{3x}$

H $(ax^2 + bx + c)e^{3x}$

18 $f(x) = 2 \sin 3x$

Options

A $a \cos 3x$

B $a \sin 3x$

C $a \cos 3x + b \sin 3x$

D $ax \cos 3x$

E $ax \sin 3x$

F $ax \cos 3x + b \sin 3x$

G $a \cos 3x + bx \sin 3x$

H $ax \cos 3x + bx \sin 3x$

Unit 7

Questions 19 to 21

The motion of an oscillating particle is given by

$$x(t) = 3 \cos \frac{1}{2} \pi t + 4 \sin \frac{1}{2} \pi t.$$

19 Select the option which gives the amplitude of the oscillations.

Options

- A 1 B 2 C 3 D 4
E 5 F 7 G 14 H 25

20 Select the option which gives the phase of the oscillations

Options

- A $\arccos(\frac{3}{5})$ B $-\arccos(\frac{3}{5})$ C $\arccos(-\frac{3}{5})$
D $-\arccos(-\frac{3}{5})$ E $\arccos(\frac{4}{5})$ F $-\arccos(\frac{4}{5})$
G $\arccos(\frac{4}{5})$ H $-\arccos(\frac{4}{5})$

21 Select the option which gives the period of the oscillations.

Options

- A $\frac{1}{2}$ B 1 C 2 D 4
E $\frac{1}{2}\pi$ F π G 2π H 4π
-

Questions 22 to 24

A particle of unit mass moves along the x -axis under the action of a force whose potential energy is

$$U(x) = 3x^2 - 8x.$$

The total mechanical energy of the particle during its motion is constant and equal to 3.

22 Select the option which is the force acting on the particle

Options

- A $6x + 8$ B $6x - 8$ C $-6x + 8$ D $-6x - 8$
E $x^3 + 4x^2$ F $x^3 - 4x^2$ G $-x^3 + 4x^2$ H $-x^3 - 4x^2$

23 Select the TWO options which give possible velocities of the particle when it is at the position $x = 1$ during its motion.

24 Select the TWO options which give values of x for which the particle is instantaneously at rest during its motion.

Options for Questions 23 and 24

- A -16 B -4 C -3 D $-\frac{1}{3}$
E $\frac{1}{3}$ F 3 G 4 H 16

[There are TWO correct options for both Questions 23 and 24.]

This assignment covers *Units 4, 6, 7 and 8*.

Note that there is no tutor-marked assignment question on *Unit 5*; this unit is assessed by computer-marked assignment questions only.

Question 1 (*Unit 4*)

This question is devoted to the development of a model of parachuting. A parachute jump consists of two parts: first a period of free fall and second a period of fall with the parachute open. It is to be assumed that in free fall the only force acting on the parachutist is her weight. However, with the parachute open the parachutist experiences another, resistive, force of magnitude kv^2 newtons, where $v \text{ m s}^{-1}$ is her speed and k is a positive constant depending on the size and construction of the parachute. The parachutist and parachute are to be modelled throughout as a particle.

To begin with, we are concerned with the period of free fall.

- (i) Supposing that initially the parachutist is at rest, find expressions for her speed
 - (a) in terms of the time t during which she has been falling,
 - (b) in terms of the distance x through which she has fallen. [4]
- (ii) If the parachutist and parachute have a total mass of 75 kg and the parachutist falls for 2 seconds before she opens her parachute, find her speed when she opens her parachute, assuming that the magnitude of the acceleration due to gravity is 10 m s^{-2} .

Consider now the period of fall with the parachute open.

- (iii) Derive, with an explanation of your reasoning, the equation of motion of the parachutist with open parachute relative to an axis directed vertically downwards with origin at the point where the parachute is opened. [5]
- (iv) Find an expression for the terminal speed v_T of the parachutist in terms of her total mass m , the constant k and the magnitude g of the acceleration due to gravity. [2]
- (v) Show that the equation of motion may be re-expressed as

$$\frac{dv}{dt} = -g \left(\frac{v^2 - v_T^2}{v_T^2} \right),$$

where t is the time during which the parachutist has been falling with her parachute open. [3]

- (vi) In free fall the parachutist gains speed very rapidly, so that when she opens her parachute she is moving with speed much bigger than v_T : indeed, it is the function of the parachute to reduce her speed to something close to v_T before she reaches the ground. Bearing this in mind, show that the solution of the differential equation in part (v) appropriate to the situation is

$$\frac{v}{v_T} = \frac{(v_0 + v_T) + (v_0 - v_T)e^{-2gt/v_T}}{(v_0 + v_T) - (v_0 - v_T)e^{-2gt/v_T}},$$

where v is the parachutist's speed when she has been falling for a time t with her parachute open, and v_0 is her speed at the instant she opens her parachute.

- (vii) If, in SI units, $m = 75$, $k = 30$ and $g = 10$, find the terminal speed v_T of the parachutist with her parachute open. Further, determine the speed of the parachutist 1 second after she opens her parachute [3]

Question 2 (Unit 6)

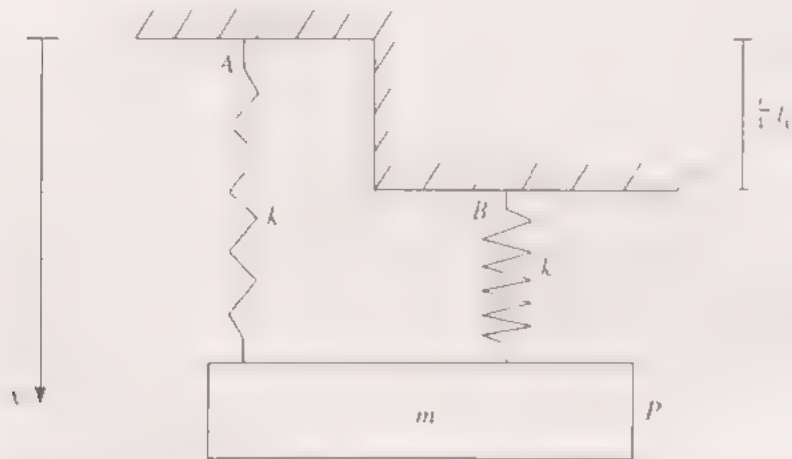
- (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{2x} + 6x + 7. \quad [17]$$

- (ii) Hence find the particular solution of the above differential equation which also satisfies the initial conditions

$$y = 0 \quad \text{and} \quad \frac{dy}{dx} = 0 \quad \text{when} \quad x = 0. \quad [8]$$

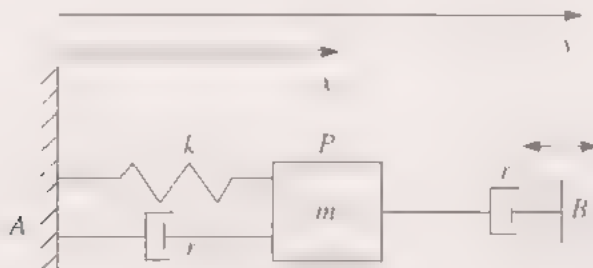
Question 3 (Unit 7)



- (i) A particle P of mass m is attached to two fixed points A and B by two identical perfect springs of stiffness k and natural length l_0 . The point A is at a height $\frac{1}{4}l_0$ above the point B. The particle is free to oscillate vertically under gravity, both springs being vertical throughout the motion, as shown in the diagram.
- (a) On a similar diagram, representing a *possible* configuration of the system during its motion, draw arrows showing the directions of the forces acting on the particle. State clearly which spring(s) you have decided to regard as extended and which as compressed. Determine the magnitudes of the forces acting on the particle for your chosen configuration when the particle P is at a distance x below the fixed point A. [6]
- (b) Derive the equation of motion of the particle. [3]
- (c) The particle's mass, the stiffness of the springs and the natural length of the springs are related by the equation $kl_0 = 4mg$, where g is the magnitude of the acceleration due to gravity. From the equation of motion obtained in part (b), find
- a general expression (containing two arbitrary constants) for the particle's position x at time t ,
 - the period of the particle's oscillations,
 - the position at which the particle can remain in equilibrium. [5]
- (ii) Consider afresh the system described in part (i) for the case $kl_0 = 4mg$.
- (a) Determine an expression for the total mechanical energy of the system when the particle P is at a distance x below the fixed point A. [5]
- (b) Initially, the particle P is released from rest a distance l_0 below the point A. By using *the conservation of energy*, find
- the speed of the particle when it is at a distance $\frac{5}{4}l_0$ below the point A,
 - the positions of the particle P when it is instantaneously at rest in its subsequent motion. [6]

Question 4 (Unit 8)

A particle P of mass m slides on a frictionless horizontal track under the influence of a perfect spring and two identical perfect dashpots. The particle is connected to a fixed point A by the spring and a dashpot, and to a forcing point B by the second dashpot, as shown in the figure.



The spring has stiffness k and natural length l_0 . The dashpot constant for each of the dashpots is r . The position of the particle P relative to the fixed point A is denoted by $x(t)$, and the position of the forcing point B relative to the fixed point A is given by $y(t)$, a known function of the time.

- (i) Obtain the differential equation for the forced vibrations of the particle relative to the fixed point A , stating clearly the configuration of the system you have assumed in your derivation.

[10]

The values of the constants, in SI units, are

$$m = 2, \quad k = 20, \quad l_0 = 5, \quad r = 4,$$

and

$$y(t) = 20 + 5 \cos 2t$$

- (ii) Show that, with these values, the equation of motion becomes

$$\ddot{x} + 4\dot{x} + 10x = 50 - 20 \sin 2t$$

2

- (iii) Find the steady-state solution of this equation *from first principles* (that is, do not simply quote a formula from the unit). Determine the amplitude and phase of the forced vibrations of the particle.

[13]

This assignment covers *Units 8, 9, 12 and 14*.

Unit 8

Questions 1 to 4

The displacement of a vibrating mechanical system is given by

$$x(t) = 2 \exp(-4t) \cos\left(3t - \frac{1}{6}\pi\right).$$

- 1 Select the option which is the damping ratio, α , of the system.
- 2 Select the option which is the undamped angular frequency, ω_0 , of the system.

Options for Questions 1 and 2

- | | | | |
|------------------------|------------------------|------------------------|------------------------|
| A 3 | B 4 | C 5 | D $\frac{3}{4}$ |
| E $\frac{3}{5}$ | F $\frac{4}{3}$ | G $\frac{4}{5}$ | H $\frac{5}{4}$ |

- 3 Select the option which is the period, τ , of the damped vibrations of the system.

Options

- | | | | |
|---------------------------|---------------------------|---------------------------|---------------------------|
| A $\frac{2\pi}{3}$ | B $\frac{3}{2\pi}$ | C $\frac{2\pi}{5}$ | D $\frac{5}{2\pi}$ |
| E $\frac{\pi}{2}$ | F $\frac{2}{\pi}$ | G π | H $\frac{1}{\pi}$ |

- 4 Select the option which is closest to the factor $x(t + \tau)/x(t)$ by which the amplitude of the oscillations decreases in each period τ .

Options

- | | | | |
|-----------------|-------------------|-------------------|--------------------|
| A 0.366 | B 0.285 | C 0.187 | D 0.123 |
| E 0.0498 | F 0.006 74 | G 0.006 56 | H 0.000 230 |

Questions 5 and 6

The equation of motion of a damped harmonic oscillator subject to an applied sinusoidal force is

$$\ddot{x} + 16\dot{x} + 100x = 100 \cos 5t.$$

- 5 Select the option which is closest to the amplitude of the steady-state vibrations of the oscillator.

Options

- | | | | |
|----------------|----------------|----------------|----------------|
| A 0.008 | B 0.010 | C 0.566 | D 0.856 |
| E 0.912 | F 1.242 | G 1.248 | H 1.314 |

- 6 Select the option which is closest to the phase of the steady-state vibrations of the oscillator, measured in radians

Options

- | | | | |
|----------------|----------------|----------------|-----------------|
| A -0.36 | B -0.82 | C -1.00 | D -1.13 |
| E -1.24 | F -1.56 | G -2.32 | H -46.98 |

Questions 7 and 8

Consider the following set of equations.

$$\begin{array}{rcl} 6x_1 + 2x_2 - x_3 & = & 5 \quad E_1 \\ -9x_1 + 7x_2 + 3x_3 & = & 6 \quad E_2 \\ -3x_1 - x_2 + 8x_3 & = & -1 \quad E_3 \end{array}$$

- 7 Select the option which describes the first step in the process of Gaussian elimination with only essential row interchanges when applied to the above set of equations.
- 8 Select the option which describes the first step in the process of Gaussian elimination with partial pivoting when applied to the above set of equations.

Options for Questions 7 and 8

- A Interchange E_1 and E_2 .
- B Interchange E_1 and E_3 .
- C Interchange E_2 and E_3 .
- D Subtract $\frac{2}{3}E_3$ from E_1 and subtract $3E_3$ from E_2 .
- E Add $\frac{2}{3}E_3$ to E_1 and add $3E_3$ to E_2 .
- F Subtract $\frac{3}{2}E_1$ from E_2 and subtract $\frac{1}{2}E_1$ from E_3 .
- G Add $\frac{3}{2}E_1$ to E_2 and add $\frac{1}{2}E_1$ to E_3 .
-

Questions 9 to 11

Three simultaneous linear equations are written in matrix form. The matrix obtained at the end of the Gaussian elimination process is

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & (k^2 - 1) & (k + 1) \end{array} \right],$$

where k is some number.

- 9 For what value or values of k do the equations have no solution?
- 10 For what value or values of k do the equations have a unique solution?
- 11 For what value or values of k do the equations have an infinite number of solutions?

Options for Questions 9 to 11

- | | |
|-------------------------------------|--|
| A All values of k | B All values of k except $k = +1$ |
| C All values of k except $k = -1$ | D All values of k except $k = \pm 1$ |
| E $k = +1$ only | F $k = -1$ only |
| G $k = \pm 1$ | H No values of k |
-

Question 12

The set of equations

$$1.0000x + 0.5000y + 0.3333z = 1.000,$$

$$0.5000x + 0.3333y + 0.2500z = 1.000,$$

$$0.3333x + 0.2500y + 0.2000z = 1.000$$

is solved by using the Gaussian elimination method with only essential row interchanges and five-significant-figure arithmetic. The solution is

$$x = 3.042, \quad y = -24.216, \quad z = 30.200.$$

Using the Gaussian elimination method with partial pivoting, the solution is

$$x = 3.042, \quad y = -24.214, \quad z = 30.198.$$

The numbers on the right-hand sides of the three equations are changed to 1.001, 0.999 and 1.001 respectively, and the equations are re-solved. Using Gaussian elimination with only essential row interchanges, the solution is

$$x = 3.118, \quad y = -24.627, \quad z = 30.593,$$

whereas using Gaussian elimination with partial pivoting, the solution is

$$x = 3.118, \quad y = -24.625, \quad z = 30.592.$$

Select the option which best describes the problem.

Options

- A The problem is ill-conditioned.
- B The problem is well-conditioned but the method of Gaussian elimination with essential row interchanges results in induced instability for this problem.
- C The problem is well-conditioned and the method of Gaussian elimination with essential row interchanges is stable for this problem.

Unit 12

Question 13

An insulated electric kettle, which has an element rated at 3 kW, contains 1.5 litres of water. Select the option which is closest to the length of time, in seconds, which it takes to heat the water from 10 °C to 80 °C. You may assume that all the electrical energy supplied goes to heating the water. The density of water is 1 kg litre⁻¹, and the specific heat of water is 4200 J kg⁻¹ °C⁻¹.

Options

- | | | | |
|------|-------|-------|-------|
| A 33 | B 38 | C 65 | D 75 |
| E 86 | F 147 | G 168 | H 441 |
-

Questions 14 to 16

Heat is transferred through a solid sandstone wall of thickness 38 cm at a rate of $2.0 \text{ W m}^{-2} \text{ }^{\circ}\text{C}^{-1}$ per unit area of wall and per unit temperature difference between its inner and outer faces. The surface area of the wall is 30 m^2 ; its inner face is at a constant uniform temperature of $20 \text{ }^{\circ}\text{C}$, and its outer face is at a constant uniform temperature of $3 \text{ }^{\circ}\text{C}$.

- 14 Select the option which is closest to the thermal conductivity of sandstone, in $\text{W m}^{-1} \text{ }^{\circ}\text{C}^{-1}$, based on these data.

Options

- | | | | | | | | |
|---|-------|---|-------|---|------|---|------|
| A | 0.045 | B | 0.067 | C | 0.33 | D | 0.76 |
| E | 1.7 | F | 4.5 | G | 6.7 | H | 7.6 |

- 15 Select the option which is closest to the magnitude of the temperature gradient, in $^{\circ}\text{C m}^{-1}$, for conduction through the wall.

Options

- | | | | | | | | |
|---|-------|---|-------|---|------|---|------|
| A | 0.017 | B | 0.022 | C | 0.45 | D | 0.61 |
| E | 1.7 | F | 2.2 | G | 45 | H | 61 |

- 16 Select the option which is closest to the rate of heat transfer through the whole wall, in watts.

Options

- | | | | | | | | |
|---|----|---|-----|---|------|---|------|
| A | 12 | B | 26 | C | 34 | D | 39 |
| E | 52 | F | 260 | G | 1000 | H | 1400 |
-

Question 17

A cylindrical copper pipe has an internal diameter of 5 cm and an external diameter of 6 cm. Its internal surface is at a constant temperature of $80 \text{ }^{\circ}\text{C}$, and its external surface is at a constant temperature of $40 \text{ }^{\circ}\text{C}$. The thermal conductivity of copper is $380 \text{ W m}^{-1} \text{ }^{\circ}\text{C}^{-1}$. Select the option which is closest to the rate of heat transfer by conduction through the wall of a 1 metre length of the pipe.

Options

- | | | | | | | | |
|---|--------|---|--------|---|---------|---|---------|
| A | 6 kW | B | 8 kW | C | 18 kW | D | 28 kW |
| E | 318 kW | F | 524 kW | G | 1206 kW | H | 1514 kW |
-

Question 18

The temperature of a radiating body is raised from $200 \text{ }^{\circ}\text{C}$ to $1200 \text{ }^{\circ}\text{C}$. Select the option which gives approximately the factor by which the heat transfer by radiation from the body is increased by the rise in temperature.

Options

- | | | | | | | | |
|---|-----|---|-----|---|-----|---|------|
| A | 3.1 | B | 6.0 | C | 9.7 | D | 30 |
| E | 36 | F | 94 | G | 220 | H | 1300 |
-

Unit 14

Questions 19 to 22

The vectors \mathbf{a} and \mathbf{b} are given in terms of the Cartesian unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} by

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k},$$

$$\mathbf{b} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

19 Select the option which is the magnitude of the vector \mathbf{a} .

20 Select the option which is the value of $\mathbf{a} \cdot \mathbf{b}$.

Options for Questions 19 and 20

A 1 B 3 C 5 D 7

E 9 F $\sqrt{3}$ G $\sqrt{5}$ H $\sqrt{7}$

21 Select the option which is the value of $3\mathbf{a} - \mathbf{b}$.

22 Select the option which is the value of $\mathbf{a} \times \mathbf{b}$.

Options for Questions 21 and 22

A $-5\mathbf{j} - 5\mathbf{k}$ B $-5\mathbf{j} - 7\mathbf{k}$ C $5\mathbf{j} - 5\mathbf{k}$

D $5\mathbf{j} - 7\mathbf{k}$ E $-4\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$ F $-4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$

G $4\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ H $4\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

Question 23

At a certain point on the Earth's surface, the vector \mathbf{a} points vertically downwards and the vector \mathbf{b} points to the North. Select the option which gives the direction of the vector $\mathbf{a} \times \mathbf{b}$.

Options

A North B South C East

D West E Vertically up F Vertically down

Question 24

Two vectors \mathbf{a} and \mathbf{b} are such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}.$$

Which one of the following options must necessarily be true?

Options

A The vectors \mathbf{a} and \mathbf{b} must both be zero.

B One or both of the vectors \mathbf{a} and \mathbf{b} must be zero.

C If \mathbf{a} and \mathbf{b} are both non-zero vectors, then \mathbf{a} and \mathbf{b} must be parallel.

D If \mathbf{a} and \mathbf{b} are both non-zero vectors, then \mathbf{a} and \mathbf{b} must be perpendicular.

E If \mathbf{a} and \mathbf{b} are both non-zero vectors, then \mathbf{a} and \mathbf{b} must both be unit vectors.

F Nothing can be concluded about the vectors \mathbf{a} and \mathbf{b} , because this relationship holds for every pair of vectors.
